$R_1, R_2, \sigma_+, \sigma_-, c$. We note that the generalized optimal structures obtained are not unique. Functions $c(x)$ and $\sigma_{ij}(x)$ can be found which have no axial symmetry, but nevertheless produce the same maximum conductivity of the ring. It can also be shown that the problem discussed here has no non-generalized optimal structures.

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ON AN INTEGRAL EQUATION FOR AXIALLY-SYMMETRIC PROBLEMS IN THE CASE OF AN ELASTIC BODY CONTAINING AN INCLUSION*

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An approximate solution of the singular integral equation (SIE) which arises in spatial problems in the theory of elasticity with mixed conditions of one-sided detachment of inclusions under axially-symmetric torsion is considered. The singularity is taken into account using the exact solution of the equation which determines the conditions of the analogous detachment in a two-dimensional problem in the case of a sheet. It is proved that, subject to certain geometric constraints, the solution of the initial problem can be obtained by the method of successive approximations. The problem of the axially-symmetric torsion of a layer using a rigid circular disc embedded in this layer and fixed to it by one of its surfaces is treated as an example. The possiblity of the practical realization of this problem lies in the fact that a torsional moment which is applied to a rod which has been welded perpendicular to the centre of a disc can be transmitted through the disc to a layer and pierce a part of the layer.

The solution of one type of singular integral equation /1/ which arises in problems on inclusions in elastic bodies which have become detached has an integrable singularity at the ends of the integration interval. In applications such as axially-symmetric problems on detached inclusions in elastic bodies /2, 3/ the need arises to construct the solution of the above-mentioned integral equation which is bounded at one of the ends of the integration interval. This solution is constructed below by the method of "large λ" /4/.

1. Let us consider a singular integral equation in the function $q(x)$

$$\pi q(x)+\int_{-1}^{1}\frac{q(\xi)\,d\xi}{\xi-x}+\frac{1}{\lambda}\int_{-1}^{1}q(\xi)\,k\left(\frac{\xi-x}{\lambda}\right)d\xi=\pi f(x) \tag{1.1}$$

$$(|x|\leqslant 1,\quad \lambda\in(0,\infty))$$

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$$k(t) = \int_0^\infty [\Lambda_1(u)\sin ut + \Lambda_2(u)\cos ut]\, du \tag{1.2}$$

The functions $\Lambda_n(z)$ $(n = 1, 2)$ are meromorphic in the plane of the complex variable $z = u + iv$ and real when $v = 0$. They have no poles on the real axis and, when $|u| \to \infty$, they satisfy the condition

$$\Lambda_n(u) = O(e^{-\varkappa_n|u|}) \quad (\varkappa_n > 0;\ n = 1, 2) \tag{1.3}$$

The properties of the functions $\Lambda_n^*(u)$ enable one to conclude that a function $k(w)$, as a function of the complex variable $w = t + i\tau$, is regular in the strip $|t| < \infty, |\tau| < \varkappa_* = \min(\varkappa_1, \varkappa_2)$ and, when $|t| < \varkappa_*$, it can be represented by the absolutely converging series /1/

$$k(t) = \sum_{n=0}^\infty b_n t^n \tag{1.4}$$

$$b_{2n} = \frac{(-1)^n}{(2n)!} \int_0^\infty \Lambda_2(u) u^{2n}\, du$$

$$b_{2n+1} = \frac{(-1)^n}{(2n+1)!} \int_0^\infty \Lambda_1(u) u^{2n+1}\, du \quad (n = 0, 1, \ldots)$$

It follows from the conditions $\max |t| = 2/\lambda$ and $|t| < \varkappa_*$ that the solution of the integral Eq.(1.1) found using the representation (1.4) can at least be used when $\lambda > \lambda_1 = 2/\varkappa_*$.

For the purpose of obtaining the solution of the singular integral Eq.(1.1) which is bounded when $x = -1$, let us consider the SIE

$$\pi q(x) + \int_{-1}^1 \frac{q(\xi)\, d\xi}{\xi - x} = \pi\chi(x) \quad (|x| \leqslant 1) \tag{1.5}$$

It has the solution /5/

$$q(x) = \frac{Q}{\pi\sqrt{2}\, X(x)} + \frac{1}{2}\chi(x) - \frac{1}{2\pi} J(x) \tag{1.6}$$

$$Q = \int_{-1}^1 q(\xi)\, d\xi, \quad J(x) = \int_{-1}^1 \frac{X(\xi)\chi(\xi)\, d\xi}{X(x)(\xi - x)}$$

$$X(x) = (1 + x)^{3/4}(1 - x)^{1/4}$$

When the condition

$$Q = \frac{1}{\sqrt{2}} \int_{-1}^1 \frac{\chi(\xi)}{Y(\xi)}\, d\xi, \quad Y(x) = \left(\frac{1+x}{1-x}\right)^{1/4} \tag{1.7}$$

is satisfied, a solution of (1.5) can be found from (1.6) which is bounded when $x = -1$. For this purpose, it is necessary to transform the integral occurring in (1.6) in the following manner:

$$J(x) = Y(x) \int_{-1}^1 \frac{\chi(\xi)\, d\xi}{Y(\xi)(\xi - x)} + \frac{1}{X(x)} \int_{-1}^1 \frac{\chi(\xi)}{Y(\xi)}\, d\xi$$

As a result of this, we have

$$q(x) = \frac{1}{2}\chi(x) - \frac{1}{2\pi} Y(x) \int_{-1}^1 \frac{\chi(\xi)\, d\xi}{Y(\xi)(\xi - x)} \tag{1.8}$$

On putting

$$\chi(x) = f(x) - \frac{1}{\pi\lambda} \int_{-1}^1 q(\xi)\, k\left(\frac{\xi - x}{\lambda}\right) d\xi$$

in (1.7) and (1.8), we get

$$q(x)=\frac{1}{2}f(x)-\frac{1}{2\pi}Y(x)\int_{-1}^{1}\frac{f(\xi)\,d\xi}{Y(\xi)(\xi-x)}+\frac{Y(x)}{2\pi^2\lambda}\int_{-1}^{1}q(\xi)F(\xi,x)\,d\xi \tag{1.9}$$

$$F(\xi,x)=-\frac{\pi}{Y(x)}k\left(\frac{\xi-x}{\lambda}\right)+\int_{-1}^{1}k\left(\frac{\xi-\eta}{\lambda}\right)\frac{d\eta}{Y(\eta)(\eta-x)}$$

$$Q=\frac{1}{\sqrt{2}}\int_{-1}^{1}\frac{f(\xi)}{Y(\xi)}d\xi-\frac{1}{\sqrt{2}\pi\lambda}\int_{-1}^{1}\frac{d\xi}{Y(\xi)}\int_{-1}^{1}q(\eta)k\left(\frac{\eta-\xi}{\lambda}\right)d\eta \tag{1.10}$$

Theorem 1. If

$$f(x)\in H_n^{\alpha}(-1,1),\ n\geqslant 0,\ {}^1/_4<\alpha\leqslant 1 \tag{1.11}$$

and a solution of the SIE (1.1) exists in $L_p(-1,1),\ p>1$ and condition (1.10) is satisfied, then the solution of this equation has the form /1/

$$q(x)=\Psi(x)Y(x),\ \Psi(x)\in C_n(-1,1) \tag{1.12}$$

for all $\lambda\in(0,\infty)$.

Lemma 1. When conditions (1.10) and (1.11) are satisfied, any solution of the SIE from the class $L_p(-1,1),\ p>1$ is also a solution of the SIE (1.9) and vice versa.

The proof of Lemma 1 is analogous to the proof of Lemma 2 in /1/.

Let us transform Eq.(1.9) to a singular integral equation in the function $\Psi(x)$:

$$\Psi=\Psi_0+A(\Psi) \tag{1.13}$$

$$\Psi_0(x)=\frac{f(x)}{2Y(x)}-\frac{1}{2\pi}\int_{-1}^{1}\frac{f(\xi)\,d\xi}{Y(\xi)(\xi-x)}$$

$$A(\Psi)\equiv\frac{1}{2\pi^2\lambda}\int_{-1}^{1}\Psi(\xi)Y(\xi)F(\xi,x)\,d\xi \tag{1.14}$$

Lemma 2. The operator A which is defined by the second formula of (1.14) acts in the space $C(-1,1)$.

The proof of this lemma is analogous to the proof of Lemma 25.2 in /4/. Here, it is necessary to transform the function $F(\xi,x)$, using the value of the integral (1.21) /1/, to the form

$$F(\xi,x)=-\pi\sqrt{2}\,k\left(\frac{\xi-x}{\lambda}\right)+I(\xi,x)$$

$$I(\xi,x)=\int_{-1}^{1}\left[k\left(\frac{\xi-\eta}{\lambda}\right)-k\left(\frac{\xi-x}{\lambda}\right)\right]\frac{d\eta}{Y(\eta)(\eta-x)}$$

Theorem 2. Let the function $f(x)\in H^{\alpha}(-1,1),\ {}^1/_4<\alpha\leqslant 1$ and the inequality

$$\lambda>\lambda_2={}^1/_4\left(D_0+\sqrt{D_0^2+4D_1}\right) \tag{1.15}$$

$$D_n=\max|k^{(n)}(t)|,\quad t\in[0,\infty]$$

hold.

In this case a solution of the SIE (1.13) in the class $C(-1,1)$ exists, it is unique and it can be found by the method of successive approximations.

For the proof of the theorem we obtain the estimate

$$|F(\xi,x)|\leqslant\pi\sqrt{2}D_0+|I(\xi,x)|<\pi\sqrt{2}D_0+\frac{\pi}{\lambda\sqrt{2}}D_1$$

which enables one to determine

$$\|A(\Psi)\|_C=\frac{1}{2\pi^2\lambda}\max_{|x|\leqslant 1}\left|\int_{-1}^{1}\Psi(\xi)Y(\xi)F(\xi,x)\,d\xi\right|<\|\Psi\|_C\frac{1}{2\lambda}\left(D_0+\frac{1}{2\lambda}D_1\right) \tag{1.16}$$

It follows from (1.16) that, when condition (1.15) is satisfied, the operator A will be a compressive operator in $C(-1,1)$.

It is convenient to determine the function $\Psi(x)$ in the form of the series

$$\Psi(x)=\sum_{n=0}^{\infty}\Psi_n(x)\lambda^{-n} \tag{1.17}$$

In order to find the functions $\Psi_n(x)$, it is necessary to introduce the expansions (1.17) and (1.4) into (1.13) and subsequently to equate the expressions accompanying the same powers of λ on the left-hand and right-hand sides of the resulting equality. As a result, we shall have

$$\Psi_1(x)=-\frac{\sqrt{2}\,b_0}{2\pi}\int_{-1}^{1}\Psi_0(\xi)\,Y(\xi)\,d\xi \tag{1.18}$$

$$\Psi_2(x)=\frac{\sqrt{2}\,b_1}{2\pi}\int_{-1}^{1}\Psi_0(\xi)\,Y(\xi)\left(x-\xi-\frac{1}{2}\right)d\xi-\frac{\sqrt{2}\,b_0}{2\pi}\int_{-1}^{1}\Psi_1(\xi)\,Y(\xi)\,d\xi$$

and so on. The function $\Psi_0(x)$ is given by the first formula of (1.14). As a result of the determination of the functions $\Psi_n(x)$ $(n=0,1,\ldots,N)$, the approximate solution of the initial SIE (1.1) will be given by the formula

$$q(x)=Y(x)\sum_{n=0}^{N}\Psi_n(x)\lambda^{-n}+O(\lambda^{-N-1}) \tag{1.19}$$

Relationship (1.19) can be used when $\lambda_*<\lambda<\infty$, where $\lambda_*=\max(\lambda_1,\lambda_2)$.

2. As an example, let us consider the axially-symmetric problem of the equilibrium of an elastic layer of thickness $2h$ in the median plane of which a thin rigid disc of radius a is located. The upper face of the small disc, when $z=+0, 0\leqslant r\leqslant a$ (r,φ,z are cylindrical coordinates) is bonded to the elastic medium while the lower face when $z=-0, 0\leqslant r\leqslant a$ is separated. The upper surface of the layer when $z=h, 0\leqslant r<\infty$ is load-free while the lower surface when $z=-h, 0\leqslant r<\infty$ is bonded to a rigid base. A torsional moment M is applied to the disc which causes the disc to rotate about the z-axis by an angle α.

This problem reduces to solving the Lamé equation for the components of the displacement vector u_φ subject to the following boundary conditions:

$$\begin{gathered}0\leqslant r\leqslant a,\ u_\varphi=-\alpha r\quad(z=+0)\\ \partial u_\varphi/\partial z=0\quad(z=-0)\\ 0\leqslant r<\infty,\ \partial u_\varphi/\partial z=0\quad(z=h)\\ u_\varphi=0\quad(z=-h)\end{gathered} \tag{2.1}$$

The problem with the boundary conditions (2.1) is reduced by the method of integral transforms to the solution of the SIE (1.1). Here,

$$\begin{gathered}f(x)=4\alpha a(x+C),\ \Lambda_1(u)=\operatorname{sch}2u\\ \Lambda_2(u)=\operatorname{th}2u-1,\ \lambda=h/a\end{gathered}$$

Here, C is a constant which has to be determined. The stresses $\tau_{\varphi z}$ in the domain where the disc is in contact with the elastic medium are expressed in terms of the function $q(x)$ by the formula

$$\tau_{\varphi z}(r,+0)=\frac{\mu}{\pi}\frac{d}{dr}\int_r^a\left[q\left(-\frac{\xi}{a}\right)-q\left(\frac{\xi}{a}\right)\right]\frac{d\xi}{\sqrt{\xi^2-r^2}}\quad(0\leqslant r\leqslant a) \tag{2.2}$$

(μ is the shear modulus). It can be shown that the structure of the solution of the SIE (1.1) which is determined by formula (1.12), ensures an integrable singularity in the case of the function $\tau_{\varphi z}$ in the form of (2.2): $\tau_{\varphi z}\sim O((a-r)^{-3/4})$ when $r\to a-0$.

A discontinuity in the displacements of the points of the cut of the elastic medium on passing from one edge of the cut to the other is also associated with the function $q(x)$:

$$u_\varphi(r,+0)-u_\varphi(r,-0)=-\frac{1}{\pi r}\int_r^a\left[q\left(\frac{\xi}{a}\right)+q\left(-\frac{\xi}{a}\right)\right]\frac{\xi d\xi}{\sqrt{\xi^2-r^2}} \tag{2.3}$$

It follows from (2.3) that the integral characteristic of the solution Q must be equated to zero. The constant C is determined from condition (1.10). The link between the parameters

M and α can be obtained from the formula

$$M = 2\pi \int_0^a r^2 \tau_{\varphi z}(r, +0)\, dr = 4\mu a^2 \int_{-1}^{1} \xi q(\xi)\, d\xi \tag{2.4}$$

(relationship (2.2) has been taken into account in deriving the latter equality).

In the case of the problem under consideration $\varkappa_1 = 2$, $\varkappa_2 = 4$, $\varkappa_* = 2$, $\lambda_1 = 1$, $D_0 = 0.347$, $D_1 = 0.460$, $\lambda_2 = 0.437$, $\lambda_* = 1$, $b_0 = -{}^1/_2 \ln 2$, $b_1 = {}^1/_2 G$ (G is the Catalan constant):

$$b_{2n} = \frac{(-1)^{n+1}}{2^{4n+1}} \left(1 - \frac{1}{2^{2n}}\right) \zeta(2n+1)$$

$$b_{2n+1} = \frac{(-1)^n}{2^{2n+1}} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^{2n+2}} \quad (n = 1, 2, \ldots)$$

and $\zeta(n)$ is the Riemann zeta-function.

By carrying out calculations using formulae (1.14) and (1.18) for the case under consideration and eliminating the constant C, we get

$$q(x) = 2\sqrt{2}\alpha a Y(x) \left[x - \frac{1}{4} + \frac{5}{16} b_2 \left(x - \frac{1}{4}\right)\lambda^{-3} - \frac{15}{32} b_3 \left(x^2 - \frac{3}{4}x - \frac{3}{16}\right)\lambda^{-4} + \frac{5}{32} b_4 \left(4x^3 - \frac{7}{2}x^2 + \frac{19}{5}x - \frac{13}{40}\right)\lambda^{-5} + O(\lambda^{-6})\right] \tag{2.5}$$

$$b_2 = 0.02817, \quad b_3 = -0.1236, \quad b_4 = -0.001899$$

By introducing $q(x)$ in the form of (2.5) into (2.4), we find the relationship between the moment M and the angle of rotation of the disc α:

$$M = \frac{5}{2}\pi\mu\alpha a^3 \left[1 + \frac{5}{16} b_2 \lambda^{-3} + \frac{15}{64} b_3 \lambda^{-4} + \frac{219}{256} b_4 \lambda^{-5} + O(\lambda^{-6})\right]$$

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